

Time-Varying Eigensystem Realization Algorithm

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DOI: 10.2514/1.45722

An identification algorithm called the time-varying eigensystem realization algorithm is proposed to realize discrete-time-varying plant models from input and output experimental data. It is shown that this singular value decomposition based method is a generalization of the eigensystem realization algorithm developed to realize time invariant models from pulse response sequences. Using the results from discrete-time identification theory, the generalized Markov parameter and the generalized Hankel matrix sequences are computed via a least squares problem associated with the input–output map. The computational procedure presented in the paper outlines a methodology to extract a state space model from the generalized Hankel matrix sequence in different time-varying coordinate systems. The concept of free response experiments is suggested to identify the subspace of the unforced system response. For the special case of systems with fixed state space dimension, the free response subspace is used to construct a uniform coordinate system for the realized models at different time steps. Numerical simulation results on general systems discuss the details and effectiveness of the algorithms.

I. Introduction

THE eigensystem realization algorithm (ERA) [1–3] has occupied the center stage in the current system identification theory and practice owing to its ease, efficiency, and robustness of implementation in several spheres of engineering. Connections of ERA with modal and principal component analyses made the algorithm an invaluable tool for the analysis of mechanical systems. As a consequence, the associated algorithms have contributed to several successful applications in design, control, and model order reduction of mechanical systems. ERA is the member of a class of algorithms derived from system realization theory based on the now classical Ho–Kalman method [4]. Because both left and right singular vector matrices of the singular value decomposition are used, ERA yields state space realizations that are not only minimal but also balanced [1]. The key utility of ERA has been in the development of discrete-time invariant models from input and output experimental data. Owing to the one-to-one mapping of linear time invariant dynamical system models between the continuous and discrete-time domains, the ERA identified discrete-time model is tantamount to the identification of a continuous-time model (with the standard assumptions on the sampling theorem). Furthermore, the physical parameters of a mechanical system (natural frequencies, normal modes, and damping) can be derived from the identified plant models by using ERA. A variety of system identification methods for such time invariant systems are available, the fundamental unifying

features of which are now well understood [5–7] and can be shown to be related (and/or equivalent) to the corresponding features of ERA.

Several efforts were undertaken in the past to develop a holistic approach for the identification of time-varying systems. Specifically, it has been desired for some time to generalize ERA to the case of time-varying systems. Earliest efforts in the development of methods for time-varying systems involved recursive and fast implementations of the time invariant methods by exploring structural properties of the input–output realizations. The classic paper by Chu et al., [8] exploring the displacement structure in the Hankel matrices is representative of the efforts of this nature. Subsequently, significant results were obtained by Shokoohi and Silverman [9] and Dewilde and Van der Veen [10], that generalized several concepts in the classical linear time invariant system theory consistently. Verhaegen and Yu [11] and Verhaegen [12] subsequently introduced the idea of repeated experiments (termed ensemble input/output data), rendering practical methods to realize the conceptual identification strategies presented earlier. These methods are referred to as ensemble state space model identification problems in the literature. This class of generalized system realization methods was applied to complex problems such as the modeling the dynamics of human joints, with much success. Liu [13] developed a methodology for developing time-varying models from free response data (for systems with an asymptotically stable origin) and made initial contributions to the development of time-varying modal parameters and their identification [14].

Although the effects of time-varying coordinate systems are shown to exist by these classical developments, it is not clear if the identified plant models (more generally identified model sequence sets) are useful in state propagation. This is because no guarantees are given as to whether the system matrices identified are, in fact, all realized in the same coordinate system. This limits the utility of the classical solutions because model sequences identified by different procedures cannot be merged as the sequences would lose compatibility at the time instance at which the algorithm is switched.

In other words, most classical results developed thus far have realized models that are topologically equivalent (defined mathematically in subsequent sections) from an input and output standpoint. However, this does not imply that they are in coordinate systems consistent in time for state propagation purposes. It is

Presented at the AAS/AIAA Spaceflight Mechanics Meeting, Galveston, TX, Jan. 2008; received 29 May 2009; revision received 16 Oct. 2009; accepted for publication 19 Oct. 2009. Copyright © 2009 by M. Majji, J. N. Juang, J. L. Junkins. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission. Copies of this paper may be made for personal or internal use, on condition that the copier pay the \$10.00 per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923; include the code 0731-5090/10 and \$10.00 in correspondence with the CCC.

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straightforward to see that the initial state given in a certain coordinate system cannot be propagated to the next time step unless the state transition and control influence matrices are expressed in the same (or compatible) coordinate system as the initial state of interest. Any misalignment would cause the state propagation to be physically meaningless and the identified plant model(s) are rendered useless.

We cannot emphasize more the importance of the coordinate transformations and their role in time-varying systems. As a practical example of this important feature underpinning the developments of this paper, let us consider the following situation. It is not too difficult to consider a version of the method proposed by Liu [13] to obtain the first few time step models. This could, in principle, be merged with the plant model sequence realized by using the classical developments of Shookohi and Silverman [9] (or equivalently Verhaegen and Yu [11]). The fact is that the plant model sequences identified appropriately in such a manner would render them incompatible at the junction (discrete-time instant) of merger, thereby making both the procedures incomplete. Looking at the facts more transparently, following Liu [13] alone, we would not have the control influence matrix sequence (one has to also observe that the formulations therein are restricted to plants with an asymptotically stable origin), and alternatively following Shookohi and Silverman [9], we would never be able to identify the first few time step (and last few time step) models because negative time indexing is not possible in general. However, following the developments of this paper, one could indeed realize the complete model sequence without invoking the negative time step experimental data or assuming asymptotic stability of the origin. Furthermore, unlike the preliminary developments of coordinate transformations by Liu [13], the solutions presented herein are compatible (give back the generalized Markov parameters indicating the arbitrariness of the transformations) and in general valid for the practical case of the number of outputs being less than the state dimension.

The methods developed in this paper, in sharp contrast with the other time-varying identification techniques mentioned earlier, arise from a perspective of generalizing the ERA to the case of time-varying systems. We develop this perspective while using the notation and preliminary developments of past researchers [9,11,13] on this problem. It is shown that the generalization thus made enables us to identify time-varying plant models that are in arbitrary coordinate systems at each time step. Furthermore, the coordinate systems at successive time steps are compatible with one another. This makes the model sequences realized useful in state propagation. The methods of computing the generalized Markov parameters using the input–output map are subsequently discussed for the two cases of the presence and absence of zero input response data in the output sequences. This is followed by a discussion on a computational procedure to determine the time-varying coordinate transformations with respect to a fixed time step t_k (most times initial condition time step) using free response experimental data. Numerical examples demonstrating the theoretical developments conclude the paper.

II. Linear Discrete-Time Varying System Realization Theory

We review the notation and definitions in linear time-varying systems following the developments presented in the classic paper by Shookohi and Silverman [9]. Linear discrete-time varying systems are governed by a set of difference equations governing the evolution of the state in time being given by

$$\mathbf{x}_{k+1} = A_k \mathbf{x}_k + B_k \mathbf{u}_k \quad (1)$$

together with a corresponding initial state vector \mathbf{x}_0 . The state variable $\mathbf{x}_k \in \mathbb{R}^n$ is most often related to the output through the measurement equation:

$$\mathbf{y}_k = C_k \mathbf{x}_k + D_k \mathbf{u}_k \quad (2)$$

with the outputs and inputs being $\mathbf{y}_k \in \mathbb{R}^m$, $\mathbf{u}_k \in \mathbb{R}^r$. Together with $A_k \in \mathbb{R}^{n \times n}$, $B_k \in \mathbb{R}^{n \times r}$, $C_k \in \mathbb{R}^{m \times n}$, and $D_k \in \mathbb{R}^{m \times r}$ being in compatible dimensions. In the following developments, it is assumed

that the true state dimension n is constant through the time period of interest. It will be transparent in the course of our developments that this assumption could be relaxed, but we retain it to facilitate some coordinate transformation results for the special class of mechanical systems important in applications. The solution of the difference equation relating the initial state and the control inputs to the state at a general time instant is given by

$$\mathbf{x}_k = \Phi(k, k_0) \mathbf{x}_0 + \sum_{j=k_0}^{k-1} \Phi(k, j+1) B_j \mathbf{u}_j \quad (3)$$

where the state transition matrix is defined in terms of its components by

$$\Phi(k, k_0) = \begin{cases} A_{k-1} A_{k-2} \dots A_{k_0}, & \forall k > k_0 \\ I, & k = k_0 \\ \text{undefined}, & \forall k < k_0 \end{cases} \quad (4)$$

Using the definition of the compound state transition matrix, the input–output relationship is given by

$$\mathbf{y}_k = C_k \Phi(k, k_0) \mathbf{x}_0 + \sum_{j=k_0}^{k-1} C_k \Phi(k, j+1) B_j \mathbf{u}_j + D_k \mathbf{u}_k \quad (5)$$

This enables us to define the input–output relationship in terms of the two index coefficients as

$$\mathbf{y}_k = C_k \Phi(k, k_0) \mathbf{x}_0 + \sum_{j=k_0}^{k-1} h_{k,j} \mathbf{u}_j + D_k \mathbf{u}_k \quad (6)$$

where the generalized Markov parameters are defined to be given by

$$h_{k,i} = \begin{cases} C_k \Phi(k, i+1) B_i, & \forall i < k-1 \\ C_k B_{k-1}, & i = k-1 \\ 0, & \forall i > k-1 \end{cases} \quad (7)$$

Similar to the time invariant case, the time-varying discrete-time systems, when expressed as the input–output map, are invariant to coordinate (similarity) transformations. In fact, the generalized Markov parameter defined earlier are invariant to a more general set of transformations called the Lyapunov transformations [9–11,15]. We briefly introduce the Lyapunov transformations in this section. We will use several notions introduced here in the subsequent sections while constructing projection maps to transform all the time-varying coordinate systems into a reference coordinate system.

In stark contrast to the time invariant (shift invariant) systems, the generalized Markov parameters determine the pulse response characteristics of the true plant in a much more general fashion. Note that the number of independent degrees of freedom to describe the input–output relationship increases tremendously for the case of time-varying systems, as the response of the system ($h_{k,i}$) not only depends upon the time difference from the applied input (\mathbf{u}_i) but also on the time instant t_i at which the said input is applied.

Following the notions set up by Shookohi and Silverman [9], the system representation $\{A_k, B_k, C_k, D_k\}$ is said to be topologically equivalent (Gohberg et al. [16] call this equivalence kinematic similarity) to the representation $\{\bar{A}_k, \bar{B}_k, \bar{C}_k, \bar{D}_k\}$ if there exists a sequence of invertible, square matrices (not necessarily related to each other, Lyapunov transformations) T_k such that $D_k = \bar{D}_k$ and

$$\bar{A}_k = T_{k+1}^{-1} A_k T_k \quad (8)$$

$$\bar{B}_k = T_{k+1}^{-1} B_k \quad (9)$$

$$\bar{C}_k = C_k T_k \quad (10)$$

It is easy to see that the compound state transition matrices have relationship similar to Eq. (8) and that all topologically equivalent representations give the same numerical value for the generalized Markov parameters owing to their definition in Eq. (7). Controllability and observability grammians are given by the infinite matrices

$$O_k := \begin{bmatrix} C_k \\ C_{k+1}A_k \\ \vdots \\ C_{k+p}A_{k+p-1} \cdots A_k \\ \vdots \end{bmatrix} \quad (11)$$

and

$$R_k = [B_k \quad A_k B_{k-1} \quad \dots] \quad (12)$$

Although the grammians are infinite matrices, usually for a system which is both controllable and observable (minimal), the principal full rank components of the corresponding grammians have most information related to the plant parameters corresponding to the current time step. This fact will enable us to construct the time-varying realizations without resorting to population of infinite matrices. The relationships between topologically equivalent representations are given by

$$\begin{aligned} \bar{O}_k &:= \begin{bmatrix} \bar{C}_k \\ \bar{C}_{k+1}\bar{A}_k \\ \vdots \\ \bar{C}_{k+p}\bar{A}_{k+p-1} \cdots \bar{A}_k \\ \vdots \end{bmatrix} = \begin{bmatrix} C_k \\ C_{k+1}A_k \\ \vdots \\ C_{k+p}A_{k+p-1} \cdots A_k \\ \vdots \end{bmatrix} T_k \\ &= O_k T_k \end{aligned} \quad (13)$$

and

$$\bar{R}_k = [\bar{B}_k \quad \bar{A}_k \bar{B}_{k-1} \quad \dots] = T_{k+1}^{-1} [B_k \quad A_k B_{k-1} \quad \dots] = T_{k+1}^{-1} R_k \quad (14)$$

Similar relations hold for block shifted controllability and observability grammians, which can be easily derived as

$$\begin{aligned} \bar{O}_k^\dagger &:= \begin{bmatrix} \bar{C}_{k+1}\bar{A}_k \\ \bar{C}_{k+2}\bar{A}_{k+1}\bar{A}_k \\ \vdots \\ \bar{C}_{k+p}\bar{A}_{k+p-1} \cdots \bar{A}_k \\ \vdots \end{bmatrix} = \begin{bmatrix} C_{k+1}A_k \\ C_{k+2}A_{k+1}A_k \\ \vdots \\ C_{k+p}A_{k+p-1} \cdots A_k \\ \vdots \end{bmatrix} T_k \\ &= O_k^\dagger T_k \end{aligned} \quad (15)$$

and

$$\begin{aligned} \bar{R}_k^\leftarrow &:= [\bar{A}_k \bar{B}_{k-1} \quad \bar{A}_k \bar{A}_{k-1} \bar{B}_{k-2} \quad \dots] \\ &= T_{k+1}^{-1} [A_k B_{k-1} \quad A_k A_{k-1} B_{k-2} \quad \dots] = T_{k+1}^{-1} R_k^\leftarrow \end{aligned} \quad (16)$$

The generalized Hankel matrix for time-varying systems is defined for every time step k to be the infinite dimensional matrix

$$\begin{aligned} H_k &= \begin{bmatrix} h_{k,k-1} & h_{k,k-2} & \cdots \\ h_{k+1,k-1} & h_{k+1,k-2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} C_k \\ C_{k+1}A_k \\ \vdots \end{bmatrix} [B_{k-1} \quad A_{k-1}B_{k-2} \quad \cdots] \\ &= O_k R_{k-1} = \bar{O}_k \bar{R}_{k-1} \end{aligned} \quad (17)$$

In general, assuming the system is uniformly observable and controllable, rank of the generalized Hankel matrix is representative of the state dimension at a given time instant. In the subsequent developments of the paper, it is assumed that the state dimension does

not change with the time index. It is not difficult to see that this assumption can be relaxed. However, we retain the assumption owing to our focus on mechanical systems, in which the connection between physical degrees of freedom and the number of state variables allows us to hold the dimensionality of the state space fixed throughout the time interval of interest. We now elaborate on the time-varying coordinate systems for discrete-time state equations and some identities governing their structure and properties.

III. Time-Varying Coordinate Systems and Transformations

As was noted from the preceding sections, the state propagation for linear time-varying systems takes place between time-varying coordinate systems. This is very similar to the concept of body-fixed rotating reference frames employed to describe rigid body rotation in attitude dynamics [16,17]. Using the notation developed thus far, consider the state propagation equations in two topologically equivalent realizations of the discrete-time-varying system. The states being propagated in the equivalent realizations are related by the time-varying transformations, $\mathbf{z}_k = T_k \mathbf{x}_k$ when the corresponding state evolution equations are written as

$$\mathbf{x}_{k+1} = A_k \mathbf{x}_k + B_k \mathbf{u}_k \quad \mathbf{y}_k = C_k \mathbf{x}_k + D_k \mathbf{u}_k \quad (18)$$

and

$$\mathbf{z}_{k+1} = \bar{A}_k \mathbf{z}_k + \bar{B}_k \mathbf{u}_k \quad \mathbf{y}_k = \bar{C}_k \mathbf{z}_k + D_k \mathbf{u}_k \quad (19)$$

Relationships between the topologically equivalent realizations presented earlier are different from the similarity transformations in case of linear time invariant systems. We rewrite the relations between these topologically equivalent realizations as

$$\bar{A}_k = T_{k+1}^{-1} A_k T_k \quad \bar{B}_k = T_{k+1}^{-1} B_k \quad \bar{C}_k = C_k T_k \quad (20)$$

The most important distinction is that the system matrices (transition matrices \bar{A}_k, A_k) do not have the same eigenvalues. Because the system evolution takes place in two different coordinate systems, T_{k+1}, T_k , this leads the basis vectors for the initial time step and the final time step to be different. Therefore, the situation is quite similar to body-fixed, rotating coordinate systems in rigid body dynamics, with the exception that the frames (basis vectors can be thought of as frames) are unknown, arbitrarily assigned by the singular value decomposition (to be discussed very shortly). A physical insight in to the coordinate systems is developed in the appendix of this paper.

The equivalent realizations, $[\bar{A}_k, A_k]$ related as in Eq. (20) in general, are not similar owing to the fact that $T_{k+1} \neq T_k$. An analyst armed with this piece of information (that the state evolution of discrete-time varying systems in general takes place between time-varying coordinate systems) is often dangerous. He/she can conclude that no physics based information can be derived from such a method because there appears to be no such information. It turns out that such a speculation is erroneous, and one can indeed extract time-varying quantities that are representative of the true time-varying system behavior from these topologically equivalent (kinematically similar) transformations. These parameters are the eigenvalues of the time-varying system matrices (true and identified) all transformed into a reference coordinate system. This is a central result of our investigation. We now detail the procedure to construct such transformations on the topologically equivalent discrete-time varying realizations Eqs. (18) and (19). Applying the general relationship, Eq. (13) between observability grammians in different coordinate systems to the realizations, Eqs. (18) and (19), we have that

$$\bar{O}_k = O_k T_k \quad (21)$$

At any other time step the same relationship holds, given by $\bar{O}_{k+p} = O_{k+p} T_{k+p}, \forall p \geq 1$. This enables us to define the quantity

$$(\bar{O}_k)^\dagger \bar{O}_{k+p} = T_k^{-1} O_k^\dagger O_{k+p} T_{k+p} \quad (22)$$

Where $\forall p \geq 1$, the identity $(\bar{O}_k)^\dagger = (O_k T_k)^\dagger = T_k^{-1} O_k^\dagger$ was used. Considering the first time step t_k , the relation between the kinematically similar system matrices is given by

$$\bar{A}_k = T_{k+1}^{-1} A_k T_k \quad (23)$$

Now we proceed to use the correction ($p = 1$) to the left of Eq. (23) and obtain a corrected system matrix \tilde{A}_k as

$$\begin{aligned} \tilde{A}_k &:= (\bar{O}_k)^\dagger \bar{O}_{k+1} (\bar{A}_k) = (T_k^{-1} O_k^\dagger O_{k+1} T_{k+1}) T_{k+1}^{-1} A_k T_k \\ &= T_k^{-1} O_k^\dagger O_{k+1} A_k T_k = T_k^{-1} \tilde{A}_k T_k \end{aligned} \quad (24)$$

where $\tilde{A}_k := O_k^\dagger O_{k+1} A_k$ is the correction employed to the time-varying system matrix in the different coordinate system. Note that both the true and identified system matrices should be transformed in to their respective reference coordinate systems. In lieu of the preceding developments, at any general time step, we have that

$$\tilde{A}_{k+p} = T_{k+p+1}^{-1} A_{k+p} T_{k+p} \quad (25)$$

In such situations we should operate on both sides to correct and obtain a transformation to the reference coordinate system (T_k in this case). This is accomplished by employing corrections on both sides given by

$$\begin{aligned} \tilde{\tilde{A}}_{k+p} &= (\bar{O}_k)^\dagger \bar{O}_{k+p+1} (\tilde{A}_{k+p}) ((\bar{O}_k)^\dagger \bar{O}_{k+p})^{-1} \\ &= T_k^{-1} O_k^\dagger O_{k+p+1} T_{k+p+1} (T_{k+p+1}^{-1} A_{k+p} T_{k+p}) (T_k^{-1} O_k^\dagger O_{k+p} T_{k+p})^{-1} \\ &= T_k^{-1} O_k^\dagger O_{k+p+1} A_{k+p} (O_k^\dagger O_{k+p})^{-1} T_k = T_k^{-1} \tilde{\tilde{A}}_{k+p} T_k \end{aligned} \quad (26)$$

The tilde ($\tilde{\cdot}$) notation has been employed to distinguish between similar and kinematically equivalent system matrices. We point out that the transformations developed earlier can also be based on the controllability grammian and are easy to derive. Note that in such a situation, however, the reference coordinate system to which the system is reset (say some Q_k) is in general independent and different from the ones obtained by using the observability grammians (denoted here by T_k). For the system identification problem, the true and identified systems are to be in kinematically similar realizations. The identified and true system needs to be transformed in order to perform a comparison. It was found that the system matrices appropriately transformed share common eigenvalues (similar system matrices). Example demonstrations illustrate this fact. The physical nature of these eigenvalues and their role in the evolution of the true system is a question that cannot be answered without performing further investigations.

IV. Time-Varying Eigensystem Realization Algorithm

We first present the algorithm to calculate the time-varying system models assuming the availability of the generalized Markov parameters. The important problem of computing the generalized Markov parameters will be addressed in the next section. A more practical algorithm for obtaining them is discussed in a paper by the authors [18] (to be published).

A. Calculation of Time-Varying Discrete-Time Models

Consider the generalized Hankel matrix populated using the generalized Markov parameters:

$$H_k^{(p,q)} = \begin{bmatrix} h_{k,k-1} & h_{k,k-2} & \cdots & h_{k,k-q} \\ h_{k+1,k-1} & h_{k+1,k-2} & \cdots & h_{k+1,k-q} \\ \vdots & \vdots & \ddots & \vdots \\ h_{k+p-1,k-1} & h_{k+p-1,k-2} & \cdots & h_{k+p-1,k-q} \end{bmatrix} \quad (27)$$

with the parameters p and q chosen such that the generalized Hankel matrix retains the rank n , the true state dimension. Insight into what numbers must be chosen is often obtained by computing the rank of the Hankel matrix. Differing ranks are possible for this generalized

time-varying Hankel matrix $H_k^{(p,q)}$ at every time step t_k for the variable state dimension problem. For problems in which the state dimension does not change, this is indicative of the validity of our assumption of a constant state dimension. This often helps the analyst in retaining appropriate numbers of row and column blocks in the Hankel matrix at a given time step for computations.

Following the identity [Eq. (17)] presented in the preceding section for the generalized Hankel matrices and using its singular value decomposition [19,20], we can write

$$H_k^{(p,q)} = O_k^{(p)} R_{k-1}^{(q)} = (U_k \Sigma_k^{1/2}) (\Sigma_k^{1/2} V_k^T) \quad (28)$$

such that expressions can be written for the corresponding controllability and observability grammians at a given time step. Notice that this decomposition is nonunique. The realizations derived from these grammians can be in any of the infinite different coordinate systems.

Now, let the controllability and observability grammians in the true (usually unknown) coordinate systems at each time step t_k be denoted by the unadorned (no superscripts) symbols, O_k , R_{k-1} , with appropriated dimensions. Then, the Hankel matrix computed observability (controllability) grammian is related to the observability (controllability) grammian in the true coordinate system by

$$O_k^{(p)} = O_k T_k \quad (R_{k-1}^{(q)} = T_k^{-1} R_{k-1}) \quad (29)$$

where again T_k is any invertible square matrix of the state dimension. Note that in problems of varying state dimension the coordinate transformations have to be appropriately redefined. Again, we do not wish to include that case in our discussions because in most mechanical system identification problems the dimensionality information can be determined a priori. This gives rise to the estimates for the system matrix as

$$O_k^{(p)\dagger} = O_k^\dagger T_k = O_{k+1} A_k T_k \quad (30)$$

where the identity $O_k^\dagger = O_{k+1} A_k$ (easily verifiable from the definitions in the preceding section) was used. However, to produce a consistent estimate, we do not have the true O_{k+1} from the decomposition of the Hankel matrix at the next time step, $H_{k+1}^{(p,q)} = O_{k+1}^{(p)} R_k^{(q)}$. But we know from the preceding developments that $O_{k+1} = O_{k+1}^{(p)} T_{k+1}^{-1}$ can be written. Substituting this expression in favor of O_k in Eq. (30), we get

$$O_k^{(p)\dagger} = O_{k+1}^{(p)} T_{k+1}^{-1} A_k T_k \quad (31)$$

This allows us to set

$$\hat{A}_k = T_{k+1}^{-1} A_k T_k = O_{k+1}^{(p)\dagger} O_k^{(p)\dagger} \quad (32)$$

as an estimate for the identified time-varying discrete system transition matrix. Notice that it is related to the unknown true system matrix but not the true system matrix. Similar estimate can be derived from the controllability grammian expressions. Considering the left shifted Hankel matrix and appropriately resetting its coordinate system, we have

$$H_{k+1}^{(p,q)\leftarrow} = O_{k+1}^{(p)} R_k^{(q)\leftarrow} \quad (33)$$

Using similar manipulations

$$R_k^{(q)\leftarrow} = T_{k+1}^{-1} R_k^\leftarrow = T_{k+1}^{-1} A_k R_{k-1} = T_{k+1}^{-1} A_k T_k R_{k-1}^{(q)} \quad (34)$$

we can set a similar estimate for the identified system matrix as

$$\hat{A}_k = R_k^{(q)\leftarrow} R_{k-1}^{(q)\dagger} \quad (35)$$

Because the first r columns of $R_k^{(q)}$ form an estimate for the identified control influence matrix, \hat{B}_k its relation to the unknown true matrix B_k is given by

$$\hat{B}_k = T_{k+1}^{-1} B_k = R_k^{(q)}(:, 1:r) \quad (36)$$

Similarly, the estimate for identified C_k is obtained by extracting the first m rows of the calculated observability grammian

$$\hat{C}_k = O_k^{(p)}(1:m, :) = C_k T_k \quad (37)$$

where the notation $M(1:m, :)$ (or $M(:, 1:r)$) denotes the first m rows (or r columns) of M matrix. Having derived the relationships between the identified and the true system models, we now proceed to the impact of the identified models in the state propagation problem.

B. State Propagation Using Identified Time-Varying Model

Let us consider any general time step t_k and the state vector in the coordinate system of the identified model be given by $\hat{\mathbf{x}}_k$. Assume that the state vector at this time step is known to be in the identified model coordinate system (i.e., $\hat{\mathbf{x}}_k = T_k^{-1} \mathbf{x}_k$ is known, whereas \mathbf{x}_k, T_k is unknown). This assumption will be relaxed shortly. Using the identified system matrices at the corresponding time step, we have

$$\hat{\mathbf{x}}_{k+1} = \hat{A}_k \hat{\mathbf{x}}_k + \hat{B}_k \mathbf{u}_k \quad \mathbf{y}_k = \hat{C}_k \hat{\mathbf{x}}_k + D_k \mathbf{u}_k \quad (38)$$

Clearly, D_k being invariant with respect to coordinate transformations, whereas the true propagation equations (had we known $\mathbf{x}_k, A_k, B_k, C_k$) are written as

$$\mathbf{x}_{k+1} = A_k \mathbf{x}_k + B_k \mathbf{u}_k \quad \mathbf{y}_k = C_k \mathbf{x}_k + D_k \mathbf{u}_k \quad (39)$$

$$\begin{aligned} \mathbf{Y} := \begin{bmatrix} \mathbf{y}_k \\ \mathbf{y}_{k+1} \\ \vdots \\ \mathbf{y}_{k+p-1} \end{bmatrix} &= \begin{bmatrix} \hat{C}_k \\ \hat{C}_{k+1} \hat{A}_k \\ \vdots \\ \hat{C}_{k+p-1} \hat{A}_{k+p-2} \dots \hat{A}_k \end{bmatrix} \hat{\mathbf{x}}_k + \begin{bmatrix} D_k \\ \hat{C}_{k+1} \hat{B}_k & D_{k+1} \\ \vdots & \vdots & \ddots \\ \hat{C}_{k+p-1} \hat{A}_{k+p-2} \dots \hat{B}_k & \hat{C}_{k+p-1} \hat{A}_{k+p-2} \dots \hat{B}_{k+1} & \dots & D_{k+p-1} \end{bmatrix} \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k+1} \\ \vdots \\ \mathbf{u}_{k+p-1} \end{bmatrix} \\ &= \hat{O}_k^{(p)} \hat{\mathbf{x}}_k + \Delta \mathbf{U} \end{aligned} \quad (44)$$

Using the derived relationships between the true and identified system matrices [Eqs. (32), (36), and (37)], we can write the Eq. (38) as

$$\hat{\mathbf{x}}_{k+1} = T_{k+1}^{-1} (A_k T_k \hat{\mathbf{x}}_k + B_k \mathbf{u}_k) \quad \mathbf{y}_k = C_k T_k \hat{\mathbf{x}}_k + D_k \mathbf{u}_k \quad (40)$$

Similarly, propagating to one more step gives us

$$\begin{aligned} \hat{\mathbf{x}}_{k+2} &= T_{k+2}^{-1} (A_{k+1} T_{k+1} \hat{\mathbf{x}}_{k+1} + B_{k+1} \mathbf{u}_{k+1}) \\ &= T_{k+2}^{-1} (A_{k+1} T_{k+1} T_{k+1}^{-1} (A_k T_k \hat{\mathbf{x}}_k + B_k \mathbf{u}_k) + B_{k+1} \mathbf{u}_{k+1}) \\ &= T_{k+2}^{-1} (A_{k+1} A_k T_k \hat{\mathbf{x}}_k + A_{k+1} B_k \mathbf{u}_k + B_{k+1} \mathbf{u}_{k+1}) \end{aligned} \quad (41)$$

Thus the state equation in general becomes (after p time steps)

$$\begin{aligned} \hat{\mathbf{x}}_{k+p} &= T_{k+p}^{-1} (A_{k+p-1} \dots A_0 T_0 \hat{\mathbf{x}}_0 + A_{k+p-1} \dots A_0 B_0 \mathbf{u}_0 + \dots \\ &\quad + A_{k+p-1} B_{k+p-2} \mathbf{u}_{k+p-2} + B_{k+p-1} \mathbf{u}_{k+p-1}) \end{aligned} \quad (42)$$

Now, considering a state propagation error defined as $e_k := T_k \hat{\mathbf{x}}_k - \mathbf{x}_k$, we have the error dynamics after $p+1$ time steps being given by the evolution equation:

$$\mathbf{e}_{p+1} = A_p A_{p-1} \dots A_0 (T_0 \hat{\mathbf{x}}_0 - \mathbf{x}_0) = A_p A_{p-1} \dots A_0 \mathbf{e}_0 \quad (43)$$

Using Lyapunov's stability theory [21] for discrete-time systems, we have that the effect of this initial coordinate system misalignment \mathbf{e}_0 will decay asymptotically to zero for time-varying systems with a stable origin (cases of asymptotic and exponential stability of the origin). However, in general, one needs to at least determine the initial conditions in the initial system coordinates

(namely, $\hat{\mathbf{x}}_0$). If one has the initial conditions in the appropriate coordinate system, the identified models can be used for state propagation.

C. Estimation of Initial Conditions from Identified Time-Varying Model

Let us now look at a method of calculating initial conditions after having identified the model matrix sequences. The question as to whether it is possible to obtain the time-varying model when the output data are inclusive of the initial condition response deserves some explanation at this point. This chicken and egg problem can be addressed in several ways. One possible solution is to write the initial condition response together with the forced response and try to solve a matrix equation relating the initial condition and input data to the sensed (noise free) outputs. This leads to a matrix equation, the solution of which, using free response data matrix is discussed in detail in the next section of the paper. Alternatively, one can use an observer based calculation as detailed in a companion manuscript developed along the lines of our recent papers [18,22]. In certain other special case situations, in which physical nature of the problem is known to the analyst, one can perform repeated experiments by physically setting the initial conditions to zero (position and velocity). In this section we concern ourselves with the problem of determination of initial conditions (in fact at a general time step t_k) after the identified plant model sequence is available.

Writing the input and output from a general k th time step, for p more time steps, one obtains a set of equations that can be written in a matrix form as

which can be solved using the least squares solution:

$$\hat{\mathbf{x}}_k^{\text{LS}} = \hat{O}_k^{(p)\dagger} (\mathbf{Y} - \Delta \mathbf{U}) \quad (45)$$

provided p is chosen sufficiently large so as to ensure the full rank of the observability grammian (= dimensionality of the state space).

D. Models for the First/Last Few Time Steps

In the problems in which time-varying model identification is of interest, it is often unclear as to how to isolate the system models for the first few time steps, in which the generalized Hankel matrix has a rank of only less than the true order of the system $\forall k, \text{rank}(H_k) < n$. The first generalized Hankel matrix in question can be written as

$$H_1^{(p,1)} = \begin{bmatrix} h_{1,0} \\ h_{2,0} \\ \vdots \\ h_{p,0} \end{bmatrix} = \begin{bmatrix} \hat{C}_1 \hat{B}_0 \\ \hat{C}_2 \hat{A}_1 \hat{B}_0 \\ \vdots \\ \hat{C}_p \hat{A}_{p-1} \dots \hat{B}_0 \end{bmatrix} \quad (46)$$

Note that it is difficult to compute the generalized Markov parameters such as $C_0 B_{-1}$ because, in practical experiments, inputs cannot be applied at negative time index so as to *feel* its response at the current time. The methodology detailed in preceding sections can be employed once a full rank Hankel matrix can be populated.

We now present a method for computing the first few time step models using an additional set of experimental data, the free response experiments. The output data of the free response experiments (also known as the zero input response) are given by

$$\begin{bmatrix} \mathbf{y}_{k,1}^{(f)} & \mathbf{y}_{k,2}^{(f)} & \cdots & \mathbf{y}_{k,N}^{(f)} \\ \mathbf{y}_{k+1,1}^{(f)} & \mathbf{y}_{k+1,2}^{(f)} & \cdots & \mathbf{y}_{k+1,N}^{(f)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_{k+p-1,1}^{(f)} & \mathbf{y}_{k+p-1,2}^{(f)} & \cdots & \mathbf{y}_{k+p-1,N}^{(f)} \end{bmatrix} = \hat{O}_k^{(p)} \hat{X}_k$$

$$= (U_k \Sigma_k^{1/2}) (\Sigma_k^{1/2} V_k^T) \quad (47)$$

where $\mathbf{y}_{k,j}^{(f)}$ for $j = 1, 2, \dots, N$ is the free decay response of the j th experiment at time k . For $\forall k = 0, 1, \dots$, Eq. (47) forms the corresponding observability grammian in the respective coordinate system as the initial conditions. Deleting the first block of data, we arrive at the block shifted output matrix that can be written as

$$\begin{bmatrix} \mathbf{y}_{k+1,1}^{(f)} & \mathbf{y}_{k+1,2}^{(f)} & \cdots & \mathbf{y}_{k+1,N}^{(f)} \\ \mathbf{y}_{k+2,1}^{(f)} & \mathbf{y}_{k+2,2}^{(f)} & \cdots & \mathbf{y}_{k+2,N}^{(f)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_{k+p,1}^{(f)} & \mathbf{y}_{k+p,2}^{(f)} & \cdots & \mathbf{y}_{k+p,N}^{(f)} \end{bmatrix} = \hat{O}_{k+1}^{(p)} \hat{X}_{k+1}$$

$$= (U_{k+1} \Sigma_{k+1}^{1/2}) (\Sigma_{k+1}^{1/2} V_{k+1}^T) \quad (48)$$

Note that the state variable ensemble at the time step t_{k+1} (denoted by \hat{X}_{k+1} with the corresponding index number $k+1$) is related to the state ensemble at time step t_k (written as \hat{X}_k) by

$$\hat{X}_{k+1} = (T_{k+1}^{-1} A_k T_k) \hat{X}_k \quad (49)$$

Using this relationship, we can derive estimates for the state transition matrix for time steps $k = 0, 1, \dots, p-1$ given by

$$\hat{A}_k := (T_{k+1})^{-1} A_k T_k = \hat{X}_{k+1} (\hat{X}_k)^{-1} \quad (50)$$

The calculation of the corresponding \hat{C}_k is accomplished by setting

$$\hat{C}_k = O_k^{(p)} (1:m, :) \quad (51)$$

The partial (rank $< n$) Hankel matrices, similar to the one written out in Eq. (46), are written for the first few time steps ($k = 0, 1, \dots, p-1$) as

$$H_{k+1}^{(p,1)} = \begin{bmatrix} h_{k+1,k} \\ h_{k+2,k} \\ \vdots \\ h_{k+p,k} \end{bmatrix} = \begin{bmatrix} \hat{C}_{k+1} \hat{B}_k \\ \hat{C}_{k+2} \hat{A}_{k+1} \hat{B}_k \\ \vdots \\ \hat{C}_{k+p} \hat{A}_{k+p-1} \cdots \hat{B}_k \end{bmatrix} \quad (52)$$

These are used in the determination of the control influence matrix as shown in the following calculation. From Eq. (52)

$$H_{k+1}^{(p,1)} = \begin{bmatrix} \hat{C}_{k+1} \hat{B}_k \\ \hat{C}_{k+2} \hat{A}_{k+1} \hat{B}_k \\ \vdots \\ \hat{C}_{k+p} \hat{A}_{k+p-1} \cdots \hat{B}_k \end{bmatrix} = O_{k+1}^{(p)} \hat{B}_k \quad (53)$$

leading to

$$\hat{B}_k = (\hat{O}_{k+1}^{(p)})^\dagger H_{k+1}^{(p,1)} \quad (54)$$

However, the model matrices determined from the Eqs. (50), (51), and (54) are of little use in practice without the coordinate transformation theory developed in Sec. III. This is because of the fact that the first few models developed in this manner are in totally different coordinate systems derived from the free response singular value decomposition. Hence, one cannot use the models thus developed in state propagation because they have a jump discontinuity at the time step $k = p$ in their coordinate systems. Using the developments of Sec. III, we correct the models by

transforming them consistently into a reference coordinate system. The transformed models are, therefore, given by

$$\tilde{A}_k := \tilde{T}_{k+1} \hat{A}_k \tilde{T}_k^{-1} = \tilde{T}_{k+1} (\hat{X}_{k+1} \hat{X}_k^{-1}) \tilde{T}_k^{-1} \quad (55)$$

$$\tilde{B}_k := \tilde{T}_{k+1} \hat{B}_k = \tilde{T}_{k+1} (\hat{O}_{k+1}^{(p)})^\dagger H_{k+1}^{(p,1)} \quad (56)$$

and

$$\tilde{C}_k := \hat{C}_k \tilde{T}_k^{-1} = O_k^{(p)} (1:m, :) \tilde{T}_k^{-1} \quad (57)$$

where the transformation (projection on to the reference \tilde{T}_k at a reference time step τ) is defined as

$$\tilde{T}_k = (\hat{O}_\tau^{(p)})^\dagger \hat{O}_k^{(p)} \quad (58)$$

Using the transformed system models and considering subsequent models in compatible coordinate systems, one therefore obtains a complete sequence of discrete-time varying models from time step $k = 0, \dots, p-1$ as long as desired by the analyst, depending on the availability of multiple experimental data. The first few models for the case of the numerical example were obtained in this manner, and the state propagation results were computed employing the transformations developed herein. Last few time step models have a dual nature, in that the system observability grammian cannot be formed fully owing to the rank deficiency of the Hankel matrix. This defect can analogously be corrected using the developments of this section.

Thus, using a framework similar to Liu [13], we arrive at different set of more general results for the first few time step models. We note in passing that there exist some structural relationships (among the generalized Markov parameters and the Hankel matrices) that lead to suggest that one can avoid repeated free response experiments. However, we could not find any useful manipulations to report at this stage and are forced to use these extra conditions to recover the first few time step models.

V. Estimation of Markov Parameters from Input–Output Data Using Least Squares Solution

As we have seen so far, the generalized Markov parameters play an important role in the determination of the time-varying plant parameters in the time-varying eigensystem realization algorithm. We now address the question as to how these Markov

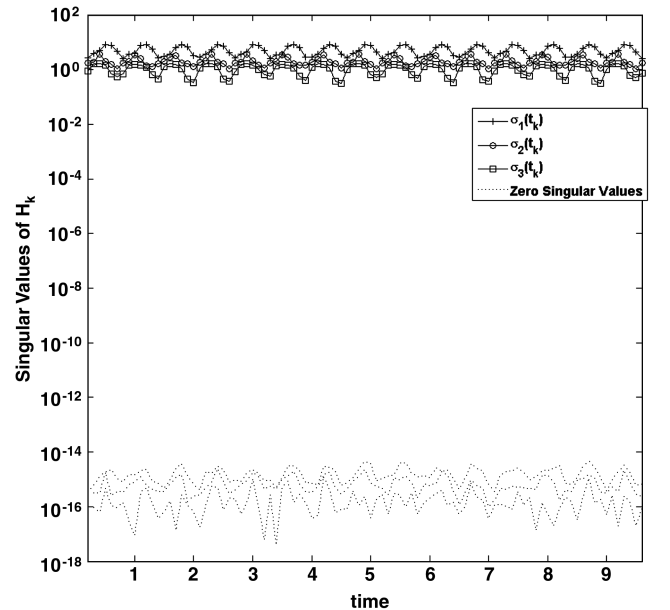


Fig. 1 Singular values of the Hankel matrix sequence, example one.

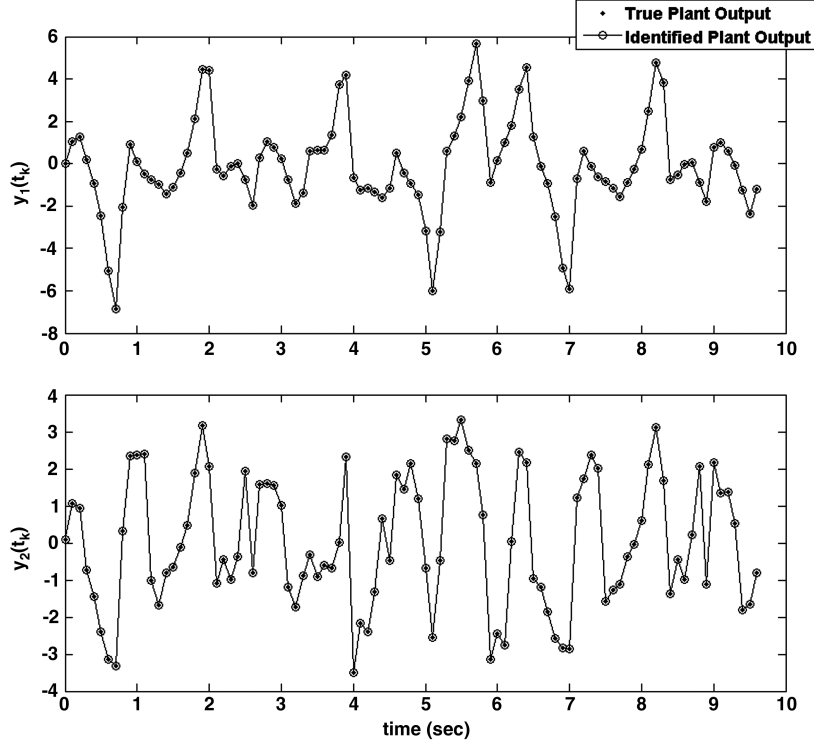


Fig. 2 Output comparison: response to test functions, example one.

parameters are computed from input–output data. For simplicity, we consider only the simple case in which the output data from multiple experiments are devoid of initial condition response. In this case, we assume that all the experiments are performed from zero initial conditions (ideal situation). In the presence of unknown initial conditions in the output data the determination of Markov parameters is more complicated because, in such a situation, one requires more information to separate out the components of the output data caused by the unknown initial conditions.

The output of the system at the time step t_k (sufficiently later than the initial time t_0) is related to the control inputs as (using Eq. (6) with $\mathbf{x}_0 = \mathbf{0}$)

$$\begin{aligned} \mathbf{y}_k &= \sum_{j=0}^{k-1} h_{k,j} \mathbf{u}_j + D_k \mathbf{u}_k = h_{k,0} \mathbf{u}_0 + \dots + h_{k,k-1} \mathbf{u}_{k-1} \\ &+ D_k \mathbf{u}_k = D_k \mathbf{u}_k + C_k B_{k-1} \mathbf{u}_{k-1} + C_k A_{k-1} B_{k-2} \mathbf{u}_{k-2} \\ &+ \dots + C_k A_{k-1} \dots A_1 B_0 \mathbf{u}_0 \end{aligned} \quad (59)$$

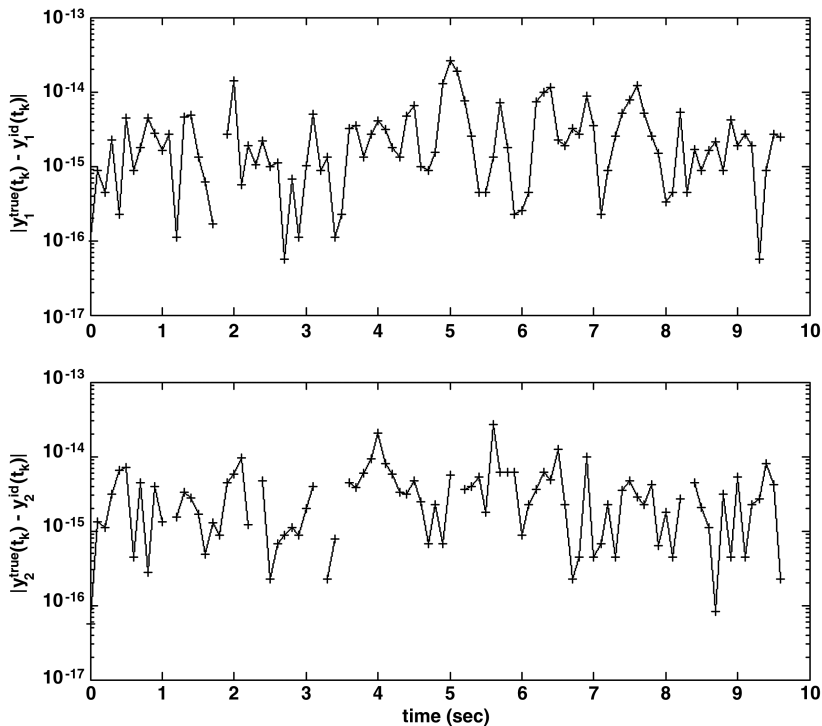


Fig. 3 Output error comparison: response to test functions, example one.

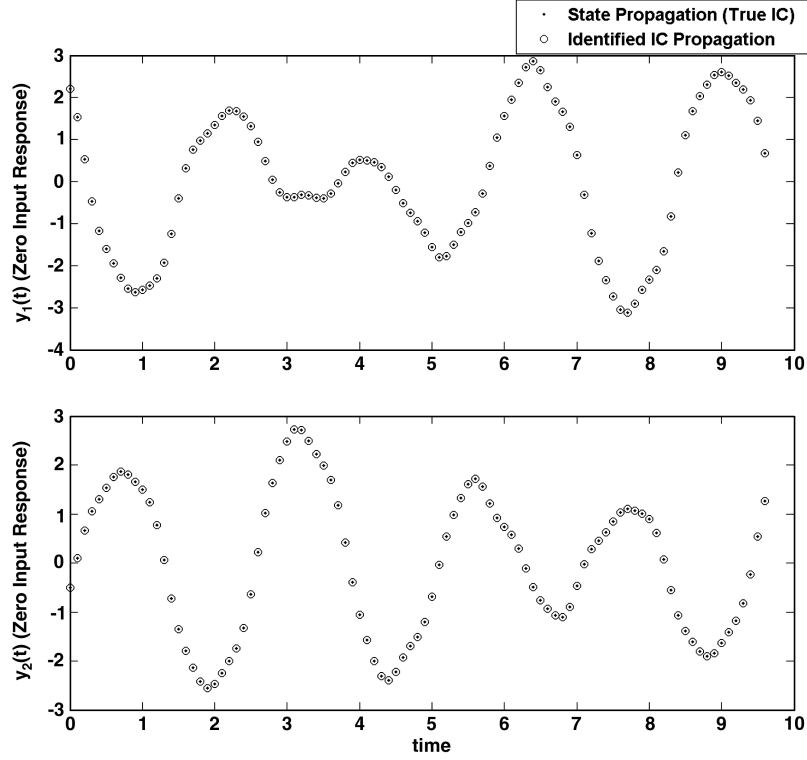


Fig. 4 Output comparison (true vs identified - initial condition determination), example two.

Stacking the generalized Markov parameters in the block matrix notation, we have that

$$\mathbf{y}_k = \begin{bmatrix} D_k & C_k B_{k-1} & \cdots & C_k A_{k-1} \cdots A_1 B_0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \vdots \\ \mathbf{u}_{k_0} \end{bmatrix} \quad (60)$$

For input-output data from multiple experiments (experiment number denoted by the superscript $(\cdot)^{(j)}$) we consequently have the matrix equation

$$\begin{bmatrix} \mathbf{y}_k^{(1)} & \mathbf{y}_k^{(2)} & \cdots & \mathbf{y}_k^{(N)} \end{bmatrix} = \begin{bmatrix} D_k & C_k B_{k-1} & \cdots & C_k A_{k-1} \cdots A_1 B_0 \end{bmatrix} \times \begin{bmatrix} \mathbf{u}_k^{(1)} & \mathbf{u}_k^{(2)} & \cdots & \mathbf{u}_k^{(N)} \\ \mathbf{u}_{k-1}^{(1)} & \mathbf{u}_{k-1}^{(2)} & \cdots & \mathbf{u}_{k-1}^{(N)} \\ \vdots & \vdots & & \vdots \\ \mathbf{u}_{k_0}^{(1)} & \mathbf{u}_{k_0}^{(2)} & \cdots & \mathbf{u}_{k_0}^{(N)} \end{bmatrix} \quad (61)$$

where the number of experiments N is chosen such that for each output time step of interest a least square solution for all the Markov parameters (until the initial time step t_0) is possible.

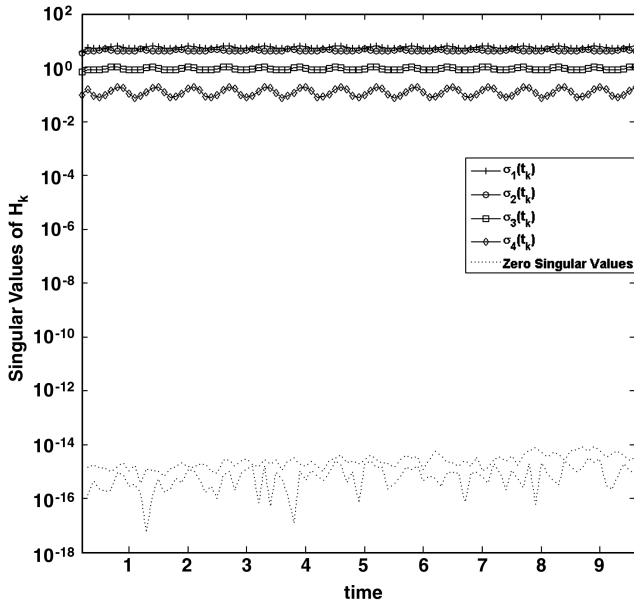


Fig. 5 Singular values of the Hankel Matrix sequence, example two.

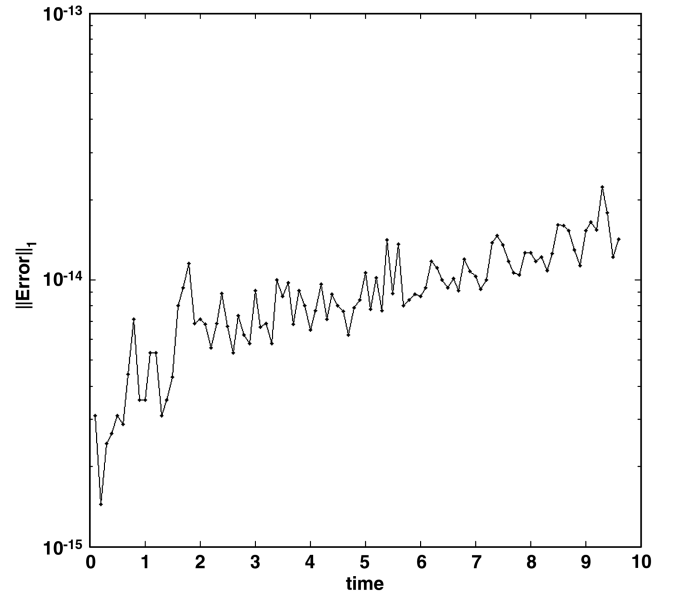


Fig. 6 Error in calculation of Markov parameters from zero state response.

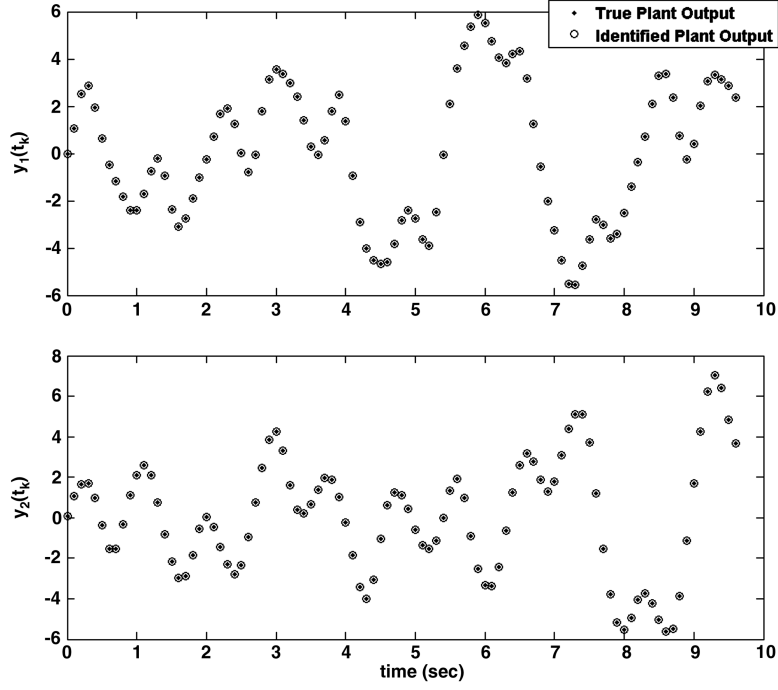


Fig. 7 Output comparison (true vs identified, forced response), example two.

The design of such increasing number of experiments is necessary to obtain a unique solution for the generalized Markov parameters from the input–output map. This increase in computations is one of the few reasons behind the lack of popularity among time-varying identification methods. In companion papers [18,22] we present techniques to remedy this increase and demonstrate the fact that the introduction of an observer in to the identification process enables a dramatic reduction of the number of required experiments while retaining the level of accuracy in the calculated generalized Markov parameters due to the existence of certain recursive relationships existing in the observer realized. These results generate sufficient optimism for the practical analyst to consider the time-varying identification methods as an alternative

in analysis and design of models for control and estimation (and/or guidance and navigation).

VI. Numerical Examples

We demonstrate the results of the paper on two representative examples. The first example has a stable origin but with true system matrix having time-varying elements that are oscillatory in nature. Second example is an oscillator example with stiffness matrix varying with time and no damping. This represents the class of problems which do not have a stable origin. We do not present examples in which the solution diverges exponentially fast to infinity as the generalized Markov parameters for such problems also go to

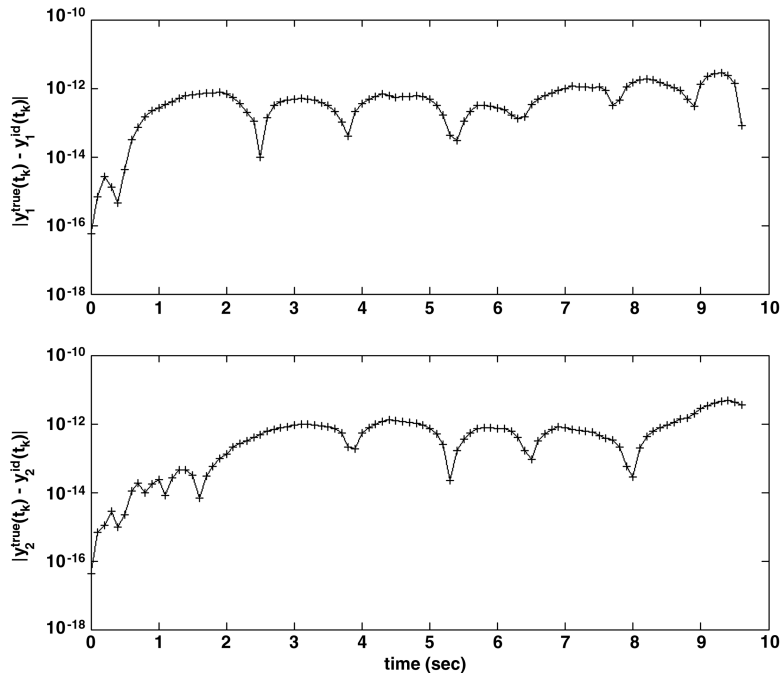


Fig. 8 Output error comparison (true vs identified, forced response), example two.

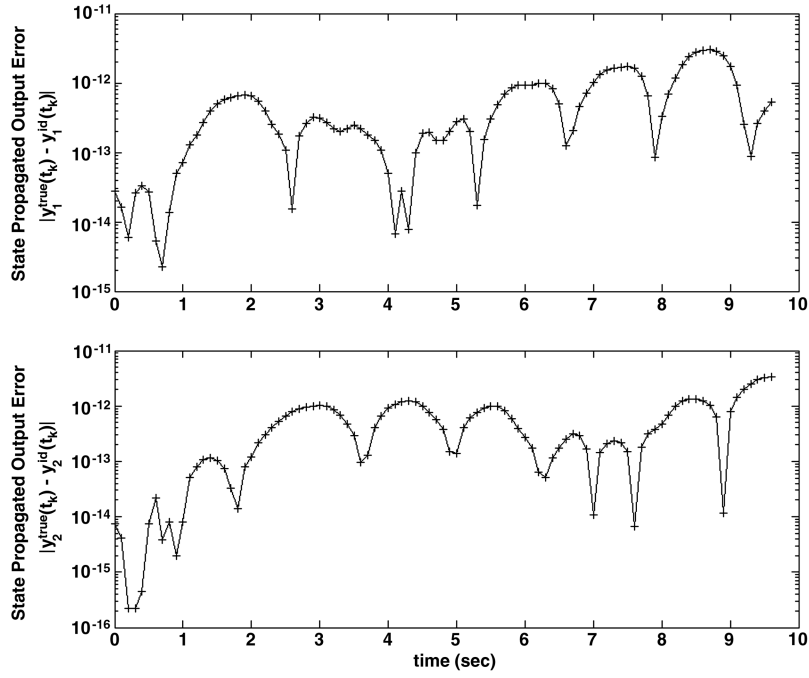


Fig. 9 Output error comparison (true vs identified, initial condition determination in the identified coordinate system), example two.

infinity and hence the input–output description becomes a highly ill-conditioned problem to facilitate any stable computations and comparisons.

A. Example One: System With a Stable Origin

Consider the time-varying system with true matrices given by

$$A_k = \begin{bmatrix} 0.3 - 0.9\tau_k & 0.1 & 0.7\tau'_k \\ 0.6\tau_k & 0.3 - 0.8\tau'_k & 0.01 \\ 0.5 & 0.15 & 0.6 - 0.9\tau_k \end{bmatrix}, \quad B_k = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \quad (62)$$

$$C_k = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad D_k = 0.1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where the time-varying elements are defined as $\tau_k = \sin(10t_k)$, $\tau'_k := \cos(10t_k)$. The first validation is performed by inspection of the rank of the Hankel matrix sequences. As the Fig. 1 clearly shows, the rank of the system remains 3 for all time indicating the order of the system, as discussed earlier in the paper. Using the least squares solution as shown in the preceding section, the system Markov parameters are determined from repeated experiments with random input signal. Identification procedure is carried out, and two test control inputs are applied to the true and identified system with zero initial conditions given by $u_1(t_k) = 0.5 \sin(12t_k)$ and $u_2(t_k) = \cos(7t_k)$. A sampling rate of 10 Hz was used in the simulation. The response for these test control inputs problem is compared in Fig. 2. Figure 3 plots the output error between the outputs of the true and identified systems to the test control input sequences.

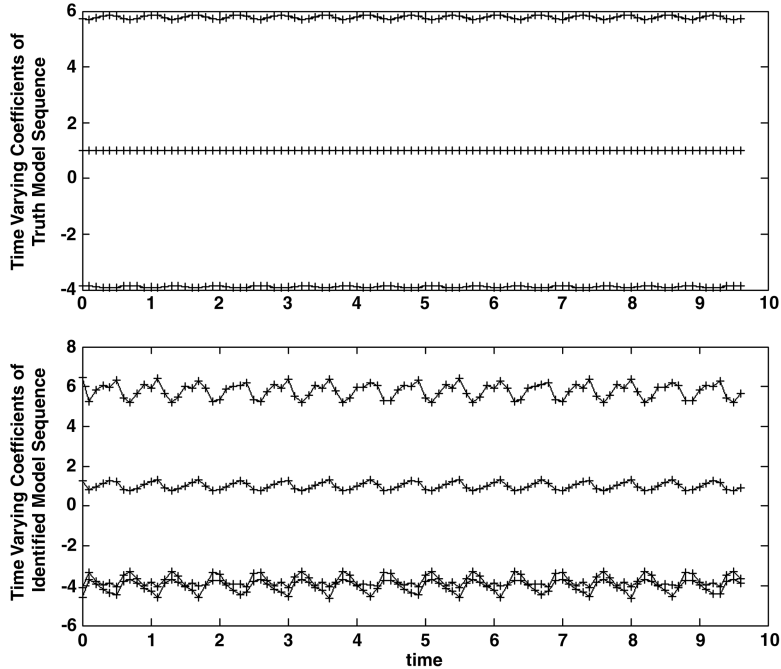


Fig. 10 Coefficients of the characteristic equation (via instantaneous time-varying companion form) of the true and identified (in time-varying coordinate systems).

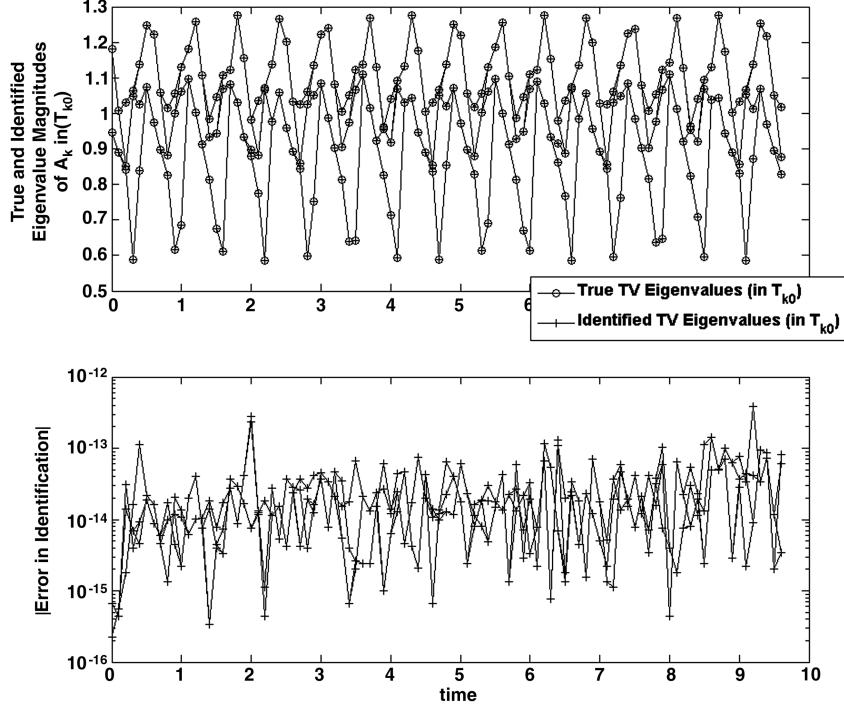


Fig. 11 Eigenvalue magnitudes of the true and identified time-varying system matrix compared in the reference coordinate system.

B. Example Two: Oscillatory System (Zero Damping)

Alternatively, we consider the system with an oscillatory nature. In this case the plant system matrix was calculated as

where the matrix is given by

$$A_c = \begin{bmatrix} 0_{2 \times 2} & I_{2 \times 2} \\ -K_t & 0_{2 \times 2} \end{bmatrix} \quad (64)$$

with

$$A_k = \exp[A_c * \Delta t] \quad B_k = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \quad C_k = \begin{bmatrix} 1 & 0 & 1 & 0.2 \\ 1 & -1 & 0 & -0.5 \end{bmatrix}, \quad D_k = 0.1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (63)$$

$$K_t = \begin{bmatrix} 4 + 3\tau_k & 1 \\ 1 & 7 + 3\tau'_k \end{bmatrix}$$

and τ_k, τ'_k are as defined in example 1. The free response of this system from true initial conditions $\mathbf{x}_0 = [1 \ 1 \ 1 \ 1]$ is plotted as the black dotted line in Fig. 4. A sampling frequency of 10 Hz was

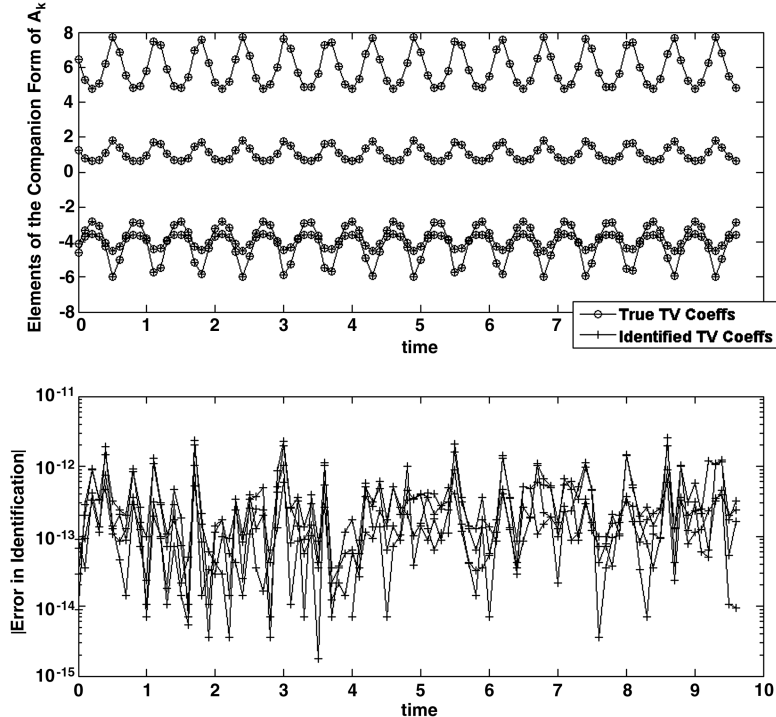


Fig. 12 Coefficients of the characteristic equation of the true and identified system (instantaneous time-varying companion form in reference coordinate system: compare with Fig. 10).

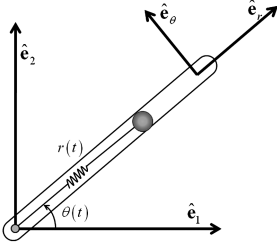


Fig. 13 Schematic depicting the point mass in a rotating tube.

employed in this simulation to delineate and discuss the results. Figure 4 shows that there is no damping inherent in the system. The singular values of the Hankel matrix sequence plotted in Fig. 5 reveal that the true order of the system is 4.

The generalized Markov parameters are determined by solving the least squares problem obtained by considering the input–output relationship as described in the preceding section. The norm of the

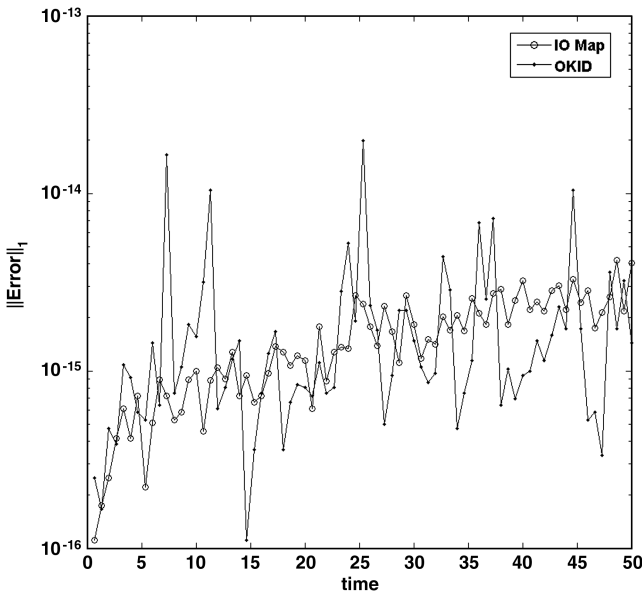


Fig. 14 Errors of the identified generalized Markov parameters (using least squares solution and the time-varying OKID procedure).

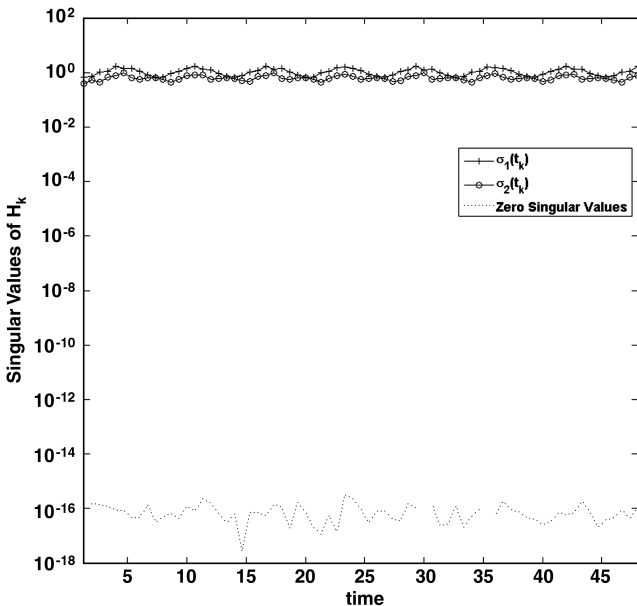


Fig. 15 Singular values of the Hankel matrix sequence (point mass in a rotating tube).

error incurred in these calculations is plotted in Fig. 6. The deterioration of the accuracy toward the end is due to the increase in the size of the least squares problem toward the end of the simulation and the deterioration of the absolute error tolerance of the solution of the linear system (for the same level of relative error maintained in the numerical solution). After identification using the Markov parameters, same test functions as the preceding example were employed and the results obtained are plotted in Fig. 7. The error between the true and the identified response to test functions is plotted in Fig. 8.

The initial condition was determined in the identified initial coordinate system using the strategy presented in Sec. IV.D. Choosing $p = 9$ in Eq. (45) for best accuracy in the normal equations, we obtained an estimate of the initial conditions (in the unknown T_0 coordinate system to be):

$$\hat{\mathbf{x}}_0 = [-1.0985 \quad -0.9383 \quad -1.7568 \quad 1.0083]^T \quad (65)$$

Using the estimated initial conditions (in their coordinate systems), the state was propagated and the free response was compared as shown in the Fig. 4. The output error between the true initial condition response and the determined initial condition response is plotted in Fig. 9.

It was found, supporting the discussions of Sec. III, that the true and identified system matrices are not similar. This is demonstrated by plotting the coefficients of the characteristic polynomial for true system and identified system in the time-varying coordinate systems in Fig. 10. Applying the transformations defined in Sec. IV, the true and identified eigenvalues (magnitude) as seen in the coordinate system T_0 [in the observable subspace at time t_0 , $O_0^{(p)}$ in Eq. (29)] are plotted as Fig. 11. The corresponding time-varying coefficients of the characteristic polynomial are shown to agree in Fig. 12.

VII. Application of the Time-Varying Identification Technique to a Problem in Dynamics

A. Problem Formulation

Consider the dynamics of a point mass in a rotating tube as shown in the schematic of Fig. 13. Dynamics of such a point mass is governed by a second order differential equation given by

$$\delta \ddot{r} = \left(\dot{\theta}^2 - \frac{k}{m} \right) \delta r + u(t) + \ell \dot{\theta}^2 \quad (66)$$

where the new variable $\delta r(t) := r(t) - \ell$ has been introduced, together with the definition of ℓ , as the free length of the spring (when no force is applied on it, i.e., Hooke's Law applies as $F_s = -k\delta r$). The function $u(t)$ is the radial control force applied on the point mass, and the parameters k , m are the spring stiffness and the mass of the point mass of interest. The time variation in this linear system is brought about by the profile of the angular velocity of the rotating tube $\dot{\theta}(t)$. Choosing the origin of the coordinate system at the position $\mathbf{r}_0 = \ell \hat{\mathbf{e}}_r$ (with no loss of generality) along the $\hat{\mathbf{e}}_r$ direction, we have the second order differential equations to be given by

$$\delta \ddot{r} = \left(\dot{\theta}^2(t) - \frac{k}{m} \right) \delta r + u(t) \quad (67)$$

where the redefinition of the origin renders the system linear time varying without any extra forcing functions.

In the first order state space form ($x_1(t) := \delta r(t)$, $x_2(t) := \delta \dot{r}(t)$) the equations can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dot{\theta}^2(t) - \frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) = A(t)\mathbf{x}(t) + B(t)u(t) \quad (68)$$

together with the measurement equations

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 0.1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \quad (69)$$

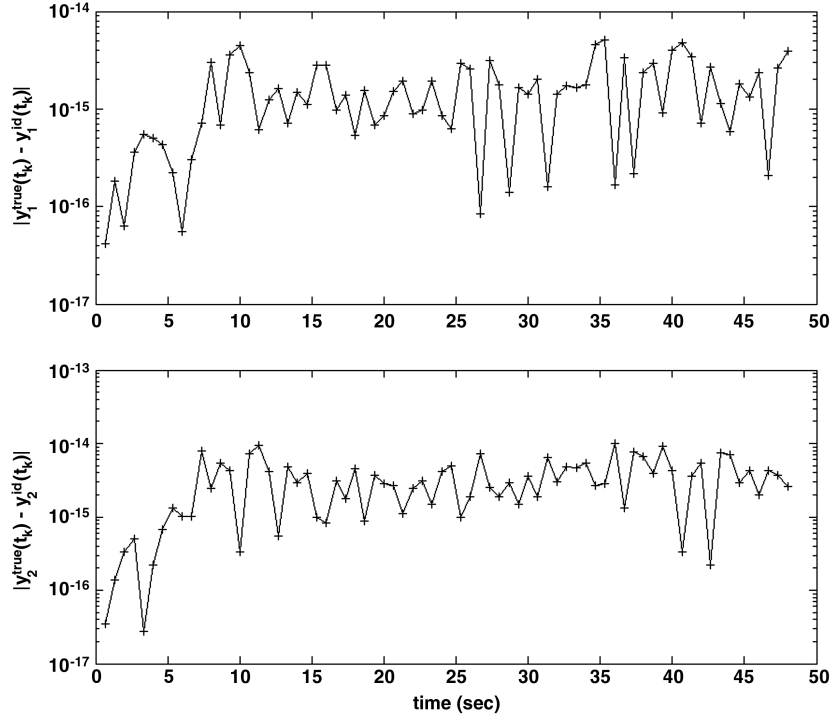


Fig. 16 Error in the identified system response.

To compare with the identified models, analytical discrete-time models were also generated by computing the state transition matrix (equivalent A_k) and the convolution integrals (equivalent B_k with a zero order hold assumption on the inputs). Because the system matrices are time varying, matrix differential equations given by

$$\dot{\Phi}(t, t_k) = A(t)\Phi(t, t_k) \quad \dot{\Psi}(t, t_k) = A(t)\Psi(t, t_k) + I \quad (70)$$

$\forall t \in [t_k, t_{k+1}]$ with initial conditions

$$\Phi(t_k, t_k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Psi(t_k, t_k) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

such that

$$A_k := \Phi(t_{k+1}, t_k) \quad B_k := \Psi(t_{k+1}, t_k)B \quad (71)$$

would represent the equivalent discrete-time varying system (truth model). Integration of the matrix differential equations was carried out with a tolerance of 1×10^{-13} (Dormand–Prince solver, subroutine ode45 of MATLAB). For the current investigation, the time variation profile of $\dot{\theta}(t) = 3 \sin(\frac{1}{2}t)$, with the mass and stiffness of the system were chosen to be $m = 1$ and $k = 10$, respectively. The time interval of interest was held to be 50 seconds, with the discretization sampling frequency set to be 1.5 Hz.

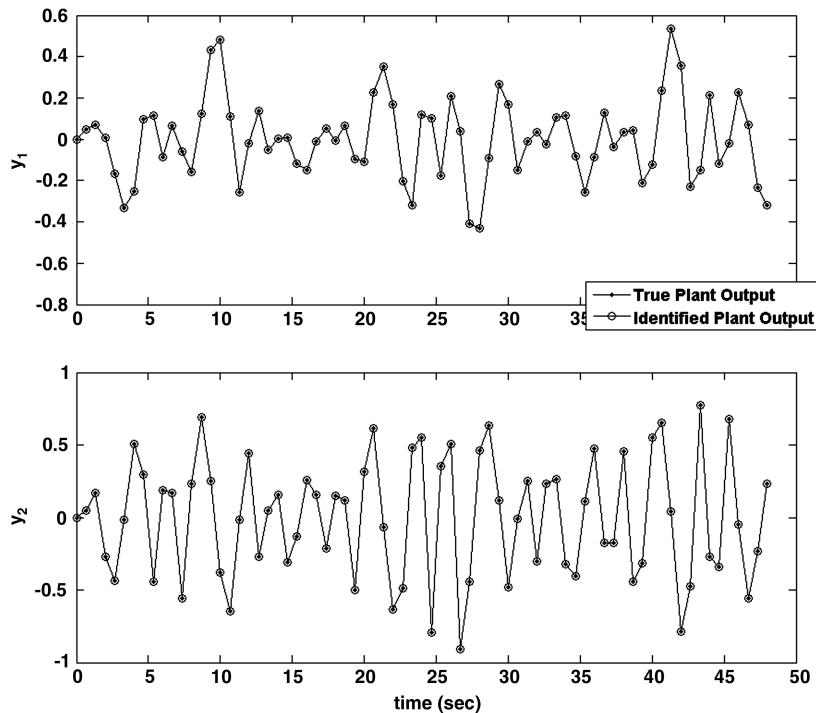


Fig. 17 Comparison of system response (true vs identified) to test control input function.

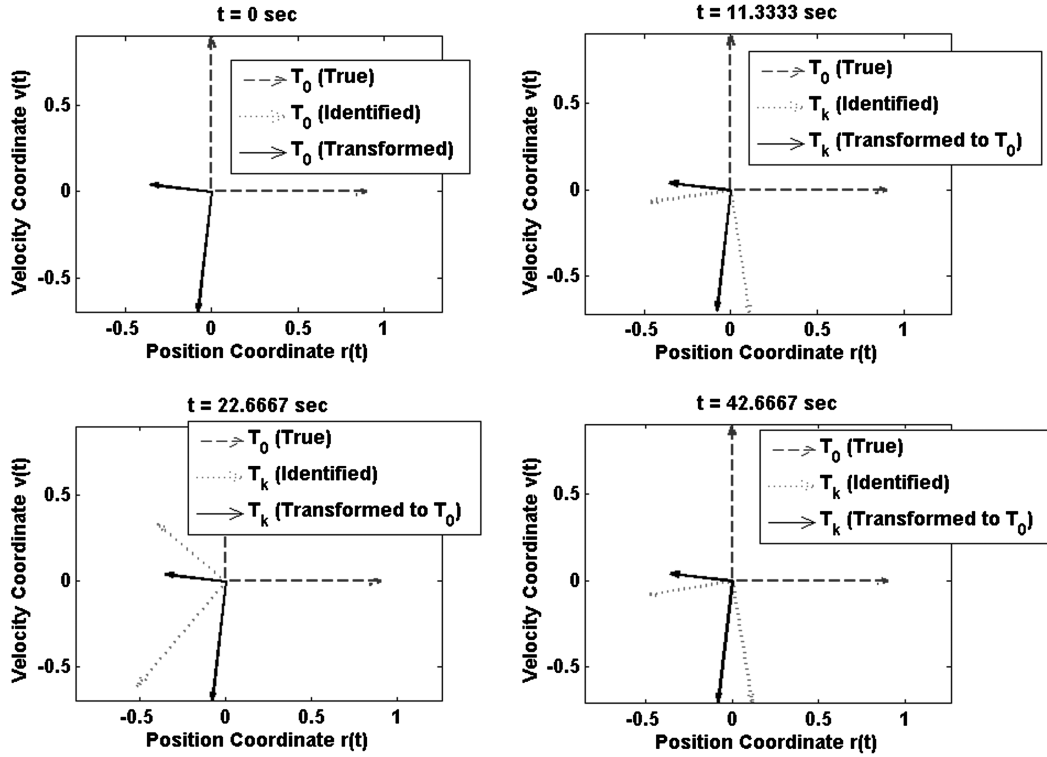


Fig. 18 Time-varying coordinate systems: graphical demonstration of the transformation process (special case, number of sensors matching the state dimension).

B. System Identification

Time-varying system identification methods developed in this paper were employed using the input-output test data from the preceding models and model sequences were obtained. Identification process starts with the determination of generalized Markov parameters, using the technique indicated in Sec. II and the time-varying observer/Kalman filter identification (TOKID) method presented in the companion manuscript [18] which minimizes the number of experiments required. Errors incurred in the determination of these Markov parameters are shown in Fig. 14. The singular values of the generalized Hankel matrix sequence are plotted in Fig. 15.

Applying a test input force $u(t) = \sin(12t)/2$ to the true and identified system matrix sequences, the error incurred in the response is shown in Fig. 16, whereas the response profiles are compared in Fig. 17. Response profiles appear jagged to show their sampled nature.

C. Discussion on the Identified Time-Varying Coordinates

This simple physical example helps us explain the time-varying coordinate systems and the transformation process. To bring further clarity into the discussion, we use the same number of sensors as the true dimensionality of the state space ($m = 2$ for this problem). Also, the generalized Hankel matrix is populated with only one redundant time step such that the observability grammians are nonredundant and hence lead to exact inverse (as opposed to pseudoinverse) in the transformations. We will first explain the transformations for this simplified situation and then proceed to a short discussion on what happens in the general situation.

For the current problem, the coordinate system calculations are simplified owing to the measurement matrix being identity as given by Eq. (69). Recall from Sec. II that this implies

$$\hat{C}_k = C_k T_k = T_k \quad (72)$$

for this problem. Considering four representative time steps, we plot the coordinate systems in Fig. 18. The long dashed arrows indicate the reference directions in the state space representing the columns of the true C_k matrix at the corresponding time step. Short dashed arrows plot the columns of the identified \hat{C}_k matrix. They represent the time-varying coordinates that are realized by the identification

algorithm. The solid line arrows represent the columns of the identified \tilde{C}_k matrix after transformed in to the reference coordinate system. A clear demonstration of the transformation process is obtained by observing that at each time step the transformed coordinates align with the reference coordinate system (at time t_0 or any other reference time step of interest).

For the more general situation of $m < n$, owing to the arbitrariness of the free basis vectors ($n - m$ of them exist at each time step), this elegant projection on to the same reference coordinate is not defined uniquely and hence the basis is completed arbitrarily (at every time step) to produce the necessary inversion (pseudoinversion to make a precise statement). The arbitrary completion of basis leads to a time-varying correction. It also depends upon the number of time steps considered for constructing the observability grammian through the least squares pseudoinverse constructed in the process of transformation. Considering different time steps would in general lead to a different transformation matrix.

VIII. Conclusions

The eigensystem realization algorithm for the identification of linear time invariant systems is extended to realize linear models that are time varying in the discrete-time domain. The time-varying extensions are derived using established notions of generalized Markov parameters and the generalized Hankel matrix sequences. It was found that the models thus realized are in different coordinate systems, inherent to the general theory of time-varying linear systems of differential (and difference) equations. It is shown that the kinematically similar (topologically equivalent) realizations are indeed similar when observed from a single reference (albeit unknown) coordinate system. This result is proposed (and used) as a tool to compare different realizations obtained by several algorithms. A method to transform the system models thus realized into a (generally unknown) reference coordinate system is presented by construction of time-varying projection operators. It is shown that the transformation matrices constructed project the realized system models into a space spanning the corresponding controllable or observable subspace at the reference time step. A method to isolate the time-varying models for the first few (and last few) time steps

using free response data from the unknown initial conditions is presented, thereby completing the sequence of models realized by the algorithm to every time step in which the experimental data are available. A least squares solution is presented for the determination of generalized Markov parameters using experimental data from repeated experiments. Numerical examples demonstrate the theoretical results of the paper. Application to a problem in dynamics provides optimism regarding the broad utility of the results derived herein, in addition to the physical insight into the time-varying system theoretic generalizations made here.

Appendix: Time-Varying Coordinate Transformations: Physical Insights

We now motivate the development of some physical insights about the time-varying coordinate systems associated with the time-varying plant model sequences that play a central role in time-varying system identification theory. It was pointed out in the Sec. III of the paper that the time-varying coordinate transformations for discrete-time varying system models are similar in nature to body-fixed coordinate systems in rigid body attitude kinematics problems.

A clear picture of this situation appears in the state transition matrices of continuous-time varying systems. To clarify this point we digress at this stage to consider the linear time-varying homogeneous system given by the continuous-time linear differential equation

$$\dot{\xi}(t) = \Lambda(t)\xi(t) \quad (\text{A1})$$

with initial conditions $\xi(t_0) = \xi_0$ and $\xi(t): \mathbb{R}^+ \rightarrow \mathbb{R}^n$, $\Lambda(t): \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then for every initial state $\xi_i(t_0)$, ($i = 1, 2, \dots, n$) spanning the state space at initial time, there exists a solution at final time $t(\neq t_0)$, denoted by $\xi_i(t)$. Collecting these solutions in to a matrix $\Psi(t) := [\xi_1(t) \ \xi_2(t) \ \dots \ \xi_n(t)]$, we arrive at the fundamental matrix [15,23]. Because it constitutes the linearly independent (arbitrary) solutions of the state differential equation, the fundamental matrix satisfies the matrix differential equation

$$\dot{\Psi}(t) = \Lambda(t)\Psi(t) \quad (\text{A2})$$

with initial conditions $\Psi(t_0) = \Psi_0 := [\xi_1(t_0) \ \xi_2(t_0) \ \dots \ \xi_n(t_0)]$ (not necessarily the $n \times n$ identity matrix).

It can be shown that this fundamental matrix is related to the state transition matrix $\Phi(t, t_0)$ as

$$\Phi(t, t_0) = \Psi(t)\Psi^{-1}(t_0) \quad (\text{A3})$$

where the classical state transition matrix [24,25] is governed by the matrix differential equation:

$$\frac{\partial}{\partial t} \Phi(t, t_0) = \Lambda(t)\Phi(t, t_0) \quad (\text{A4})$$

with initial conditions $\Phi(t, t_0) = I_n$ as the unit (identity) matrix. Realizing that the solution structure at any time t is given by $\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}_0$ we point out the stark contrast to time invariant system ($\Lambda(t) = A_c$), where the state transition matrix is given by $\Phi(t, t_0) = \exp[A_c(t - t_0)]$. Note that the solution in the time invariant case remains in the same space owing to the power series expansion definition of the matrix exponential. That is the space spanned by $(I, A_c, A_c^2, \dots)\mathbf{x}_0$. Such a definition/parallelism cannot be made for time-varying systems and hence the state transition matrix, as given by Eq. (A3), maps the state in one coordinate system at initial time step t_0 to a possibly (usually) different coordinate system at any subsequent time t . Therefore, it emerges conclusively that the true and identified system matrices in our current discussions are special instances of the fundamental matrices outlined in Eq. (A2) with an arbitrary set of basis functions at the corresponding initial time step. This simple observation is evidently new and of fundamental importance in establishing a complete system identification algorithm for time-varying systems. The ideas presented in the section are graphically illustrated in Fig. A1, in which a 3-D state space is assumed for clarity in demonstration.

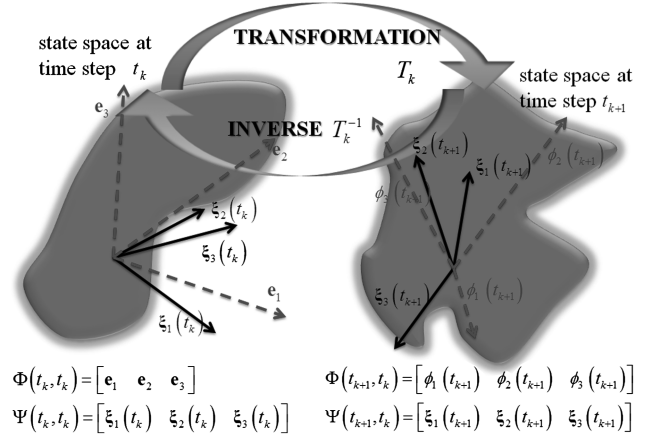


Fig. A1 Illustration of time-varying coordinate systems and transformations.

Acknowledgments

The authors wish to acknowledge the support of Texas Institute of Intelligent Bio-Nano Materials and Structures for Aerospace Vehicles funded by NASA Cooperative Agreement No. NCC-1-02038. Any opinions, findings and conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NASA. John E. Hurtado is warmly acknowledged for suggesting the example application problem.

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